

Indefinite Problems

Timothy Daly, Gilbert Baumslag, William Sit, Sean Cleary, Douglas Troeger

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Abstract

An initial set of thoughts on the question of using indefinites in Axiom.
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1 Polynomials

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

where

$$\binom{n}{k} = {}_n C_k = \frac{n!}{(n-k)!k!}$$

so

$$(x + y)^8 = {}_8 C_0 x^{8-0} y^0 + {}_8 C_1 x^{8-1} y^1 + {}_8 C_2 x^{8-2} y^2 + {}_8 C_3 x^{8-3} y^3 + {}_8 C_4 x^{8-4} y^4 + {}_8 C_5 x^{8-5} y^5 + {}_8 C_6 x^{8-6} y^6 + {}_8 C_7 x^{8-7} y^7 + {}_8 C_8 x^{8-8} y^8$$

1.1 particular cases

1.2 step 1 $(m^0) * (n^i)$

The point of this exercise is to find general forms for particular polynomials. In particular, as above, the binomial form has a convenient closed form. So we're looking for a closed form of the general product

$$(x + a)^n * (x + b)^m$$

Let $m = (x + a)$ and $m = (x + b)$, we examine particular cases:

$$\begin{aligned} (m^0) * (n^0) &= b^0 x^0 \\ (m^0) * (n^1) &= b^0 x^1 + b^1 x^0 \\ (m^0) * (n^2) &= b^0 x^2 + 2b^1 x^1 + b^2 x^0 \\ (m^0) * (n^3) &= b^0 x^3 + 3b^1 x^2 + 3b^2 x^1 + b^3 x^0 \\ (m^0) * (n^4) &= b^0 x^4 + 4b^1 x^3 + 6b^2 x^2 + 4b^3 x^1 + b^4 x^0 \\ (m^0) * (n^5) &= b^0 x^5 + 5b^1 x^4 + 10b^2 x^3 + 10b^3 x^2 + 5b^4 x^1 + b^5 x^0 \end{aligned}$$

The first observation is that the coefficients follow pascal's triangle (hardly a surprise) and could be rewritten as:

$$\begin{aligned} (m^0) * (n^0) &= {}_0 C_0 b^0 x^0 \\ (m^0) * (n^1) &= {}_1 C_0 b^0 x^1 + {}_1 C_1 b^1 x^0 \\ (m^0) * (n^2) &= {}_2 C_0 b^0 x^2 + {}_2 C_1 b^1 x^1 + {}_2 C_2 b^2 x^0 \\ (m^0) * (n^3) &= {}_3 C_0 b^0 x^3 + {}_3 C_1 b^1 x^2 + {}_3 C_2 b^2 x^1 + {}_3 C_3 b^3 x^0 \\ (m^0) * (n^4) &= {}_4 C_0 b^0 x^4 + {}_4 C_1 b^1 x^3 + {}_4 C_2 b^2 x^2 + {}_4 C_3 b^3 x^1 + {}_4 C_4 b^4 x^0 \\ (m^0) * (n^5) &= {}_5 C_0 b^0 x^5 + {}_5 C_1 b^1 x^4 + {}_5 C_2 b^2 x^3 + {}_5 C_3 b^3 x^2 + {}_5 C_4 b^4 x^1 + {}_5 C_5 b^5 x^0 \end{aligned}$$

and if we observe the behavior of the b term we can rewrite it as

$$\begin{aligned} (m^0) * (n^0) &= \sum_{i=0}^0 {}_0 C_i b^i x^0 \\ (m^0) * (n^1) &= \sum_{i=0}^1 {}_1 C_i b^i x^i \\ (m^0) * (n^2) &= \sum_{i=0}^2 {}_2 C_i b^i x^i \\ (m^0) * (n^3) &= \sum_{i=0}^3 {}_3 C_i b^i x^i \\ (m^0) * (n^4) &= \sum_{i=0}^4 {}_4 C_i b^i x^i \\ (m^0) * (n^5) &= \sum_{i=0}^5 {}_5 C_i b^i x^i \end{aligned}$$

or in functional terms we can define a `term` function which will generate each term in the polynomial as:

$$\text{term}(\alpha, \beta) = {}_{\alpha}C_{\beta} b^{\beta} x^{\alpha-\beta}$$

So, the Axiom code is: `term(alpha,beta)== pascalTriangle(alpha,beta)$GALUTIL(INT)*(b**(beta))*(x**(alpha-beta))` and we can compute particular terms thus:

`term(0,0)`

$$1$$

`term(1,0)`

$$x$$

`term(1,1)`

$$b$$

and the polynomial is just

`term(1,0)+term(1,1)`

$$x + b$$

with the next polynomial:

`term(2,0)+term(2,1)+term(2,2)`

$$x^2 + 2 b x + b^2$$

Thus the more general conclusion of

$$(n^k) = \sum_{i=0}^k \text{term}(k, i)$$

which in math notation is:

$$(n^k) = \sum_{i=0}^k {}_kC_i b^i x^{k-i}$$

1.3 step 2 $(m^1) * (n^i)$

$$\begin{aligned}
(m^1) * (n^0) &= x^1 + ax^0 \\
(m^1) * (n^1) &= x^2 + (b+a)x^1 + abx^0 \\
(m^1) * (n^2) &= x^3 + (2b+a)x^2 + (b^2+2ab)x^1 + ab^2x^0 \\
(m^1) * (n^3) &= x^4 + (3b+a)x^3 + (3b^2+3ab)x^2 + (b^3+3ab^2)x^1 + ab^3x^0 \\
(m^1) * (n^4) &= x^5 + (4b+a)x^4 + (6b^2+4ab)x^3 + (4b^3+6ab^2)x^2 \\
&\quad + (b^4+4ab^3)x^1 + ab^4x^0 \\
(m^1) * (n^5) &= x^6 + (5b+a)x^5 + (10b^2+5ab)x^4 \\
&\quad + (10b^3+10ab^2)x^3 + (5b^4+10ab^3)x^2 + (b^5+5ab^4)x^1 + ab^5x^0
\end{aligned}$$

1.4 step 3 $(m^2) * (n^i)$

$$\begin{aligned}
(m^2) * (n^0) &= x^2 + 2ax^1 + a^2x^0 \\
(m^2) * (n^1) &= x^3 + (b+2a)x^2 + (2ab+a^2)x + abx^0 \\
(m^2) * (n^2) &= x^4 + (2b+2a)x^3 + (b^2+4ab+a^2)x^2 \\
&\quad + (2ab^2+2a^2b)x^1 + a^2b^2x^0 \\
(m^2) * (n^3) &= x^5 + (3b+2a)x^4 + (3b^2+6ab+a^2)x^3 + \\
&\quad (b^3+6ab^2+3a^2b)x^2 + (2ab^3+3a^2b^2)x^1 + a^2b^3x^0 \\
(m^2) * (n^4) &= x^6 + (4b+2a)x^5 + (6b^2+8ab+a^2)x^4 + \\
&\quad (4b^3+12ab^2+4a^2b)x^3 + (b^4+8ab^3+6a^2b^2)x^2 \\
&\quad + (2ab^4+4a^2b^3)x^1 + a^2b^4x^0 \\
(m^2) * (n^5) &= x^7 + (5b+2a)x^6 + (10b^2+10ab+a^2)x^5 \\
&\quad + (10b^3+20ab^2+5a^2b)x^4 + (5b^4+20ab^3+10a^2b^2)x^3 \\
&\quad + (b^5+10ab^4+10a^2b^3)x^2 + (2ab^5+5a^2b^4)x^1 + a^2b^5x^0
\end{aligned}$$

1.5 step 4 $(m^3) * (n^i)$

$$\begin{aligned}
(m^3) * (n^0) &= x^3 + 3ax^2 + 3a^2x^1 + a^3x^0 \\
(m^3) * (n^1) &= x^4 + (b+3a)x^3 + (3ab+3a^2)x^2 + (3a^2b+a^3)x^1 + a^3bx^0 \\
(m^3) * (n^2) &= x^5 + (2b+3a)x^4 + (b^2+6ab+3a^2)x^3 \\
&\quad + (3ab^2+6a^2b+a^3)x^2 + (3a^2b^2+2a^3b)x^1 + a^3b^2x^0 \\
(m^3) * (n^3) &= x^6 + (3b+3a)x^5 + (3b^2+9ab+3a^2)x^4 + \\
&\quad (b^3+9ab^2+9a^2b+a^3)x^3 + (3ab^3+9a^2b^2+3a^3b)x^2 \\
&\quad + (3a^2b^3+3a^3b^2)x^1 + a^3b^3x^0 \\
(m^3) * (n^4) &= x^7 + (4b+3a)x^6 + (6b^2+12ab+3a^2)x^5 \\
&\quad + (4b^3+18ab^2+12a^2b+a^3)x^4 + (b^4+12ab^3+18a^2b^2+4a^3b)x^3 \\
&\quad + (3ab^4+12a^2b^3+6a^3b^2)x^2 + (3a^2b^4+4a^3b^3)x^1 + a^3b^4x^0 \\
(m^3) * (n^5) &= x^8 + (5b+3a)x^7 + (10b^2+15ab+3a^2)x^6 \\
&\quad + (10b^3+30ab^2+15a^2b+a^3)x^5 + (5b^4+30ab^3+30a^2b^2+5a^3b)x^4 \\
&\quad + (b^5+15ab^4+30a^2b^3+10a^3b^2)x^3 + (3ab^5+15a^2b^4+10a^3b^3)x^2 \\
&\quad + (3a^2b^5+5a^3b^4)x^1 + a^3b^5x^0
\end{aligned}$$

1.6 step 5 $(m^4) * (n^i)$

$$\begin{aligned}
 (m^4) * (n^0) &= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x^1 + a^4x^0 \\
 (m^4) * (n^1) &= x^5 + (b + 4a)x^4 + (4ab + 6a^2)x^3 + (6a^2b + 4a^3)x^2 \\
 &\quad + (4a^3b + a^4)x^1 + a^4bx^0 \\
 (m^4) * (n^2) &= x^6 + (2b + 4a)x^5 + (b^2 + 8ab + 6a^2)x^4 + (4ab^2 + 12a^2b + 4a^3)x^3 \\
 &\quad + (6a^2b^2 + 8a^3b + a^4)x^2 + (4a^3b^2 + 2a^4b)x^1 + a^4b^2x^0 \\
 (m^4) * (n^3) &= x^7 + (3b + 4a)x^6 + (3b^2 + 12ab + 6a^2)x^5 + \\
 &\quad (b^3 + 12ab^2 + 18a^2b + 4a^3)x^4 + (4ab^3 + 18a^2b^2 + 12a^3b + a^4)x^3 \\
 &\quad + (6a^2b^3 + 12a^3b^2 + 3a^4b)x^2 + (4a^3b^3 + 3a^4b^2)x^1 + a^4b^3x^0 \\
 (m^4) * (n^4) &= x^8 + (4b + 4a)x^7 + (6b + 16ab + 6a)x^6 + (4b + 24ab + 24ab + 4a)x^5 \\
 &\quad + (b + 16ab + 36ab + 16ab + a)x^4 + (4ab + 24ab + 24ab + 4ab)x^3 \\
 &\quad + (6ab + 16ab + 6ab)x^2 + (4ab + 4ab)x^1 + abx^0 \\
 (m^4) * (n^5) &= x^9 + (5b + 4a)x^8 + (10b^2 + 20ab + 6a^2)x^7 \\
 &\quad + (10b^3 + 40ab^2 + 30a^2b + 4a^3)x^6 \\
 &\quad + (5b^4 + 40ab^3 + 60a^2b^2 + 20a^3b + a^4)x^5 \\
 &\quad + (b^5 + 20ab^4 + 60a^2b^3 + 40a^3b^2 + 5a^4b)x^4 \\
 &\quad + (4ab^5 + 30a^2b^4 + 40a^3b^3 + 10a^4b^2)x^3 \\
 &\quad + (6a^2b^5 + 20a^3b^4 + 10a^4b^3)x^2 \\
 &\quad + (4a^3b^5 + 5a^4b^4)x^1 + a^4b^5x^0
 \end{aligned}$$

which could be more insightfully written as

$$\begin{aligned}
(m^4) * (n^0) &= && x^4 \\
&&& +4ax^3 \\
&&& +6a^2x^2 \\
&&& +4a^3x^1 \\
&&& +a^4x^0 \\
(m^4) * (n^1) &= && x^5 \\
&&& +(b+4a)x^4 \\
&&& +(4ab+6a^2)x^3 \\
&&& +(6a^2b+4a^3)x^2 \\
&&& +(4a^3b+a^4)x^1 \\
&&& +a^4bx^0 \\
(m^4) * (n^2) &= && x^6 \\
&&& +(2b+4a)x^5 \\
&&& +(b^2+8ab+6a^2)x^4 \\
&&& +(4ab^2+12a^2b+4a^3)x^3 \\
&&& +(6a^2b^2+8a^3b+a^4)x^2 \\
&&& +(4a^3b^2+2a^4b)x^1 \\
&&& +a^4b^2x^0 \\
(m^4) * (n^3) &= && x^7 \\
&&& +(3b+4a)x^6 \\
&&& +(3b^2+12ab+6a^2)x^5 \\
&&& +(b^3+12ab^2+18a^2b+4a^3)x^4 \\
&&& +(4ab^3+18a^2b^2+12a^3b+a^4)x^3 \\
&&& +(6a^2b^3+12a^3b^2+3a^4b)x^2 \\
&&& +(4a^3b^3+3a^4b^2)x^1 \\
&&& +a^4b^3x^0 \\
(m^4) * (n^4) &= && x^8 \\
&&& +(4b+4a)x^7 \\
&&& +(6b+16ab+6a)x^6 \\
&&& +(4b+24ab+24ab+4a)x^5 \\
&&& +(b+16ab+36ab+16ab+a)x^4 \\
&&& +(4ab+24ab+24ab+4ab)x^3 \\
&&& +(6ab+16ab+6ab)x^2 \\
&&& +(4ab+4ab)x^1 \\
&&& +abx^0 \\
(m^4) * (n^5) &= && x^9 \\
&&& +(5b+4a)x^8 \\
&&& +(10b^2+20ab+6a^2)x^7 \\
&&& +(10b^3+40ab^2+30a^2b+4a^3)x^6 \\
&&& +(5b^4+40ab^3+60a^2b^2+20a^3b+a^4)x^5 \\
&&& +(b^5+20ab^4+60a^2b^3+40a^3b^2+5a^4b)x^4 \\
&&& +(4ab^5+30a^2b^4+40a^3b^3+10a^4b^2)x^3 \\
&&& +(6a^2b^5+20a^3b^4+10a^4b^3)x^2 \\
&&& +(4a^3b^5+5a^4b^4)x^1 \\
&&& +a^4b^5x^0
\end{aligned}$$

which could be more insightfully written as

$$\begin{aligned}
(m^4) * (n^0) &= && x^4 \\
&&& +4ax^3 \\
&&& +6a^2x^2 \\
&&& +4a^3x^1 \\
&&& +a^4x^0 \\
(m^4) * (n^1) &= && x^5 \\
&&& +(b+4a)x^4 \\
&&& +(4ab+6a^2)x^3 \\
&&& +(6a^2b+4a^3)x^2 \\
&&& +(4a^3b+a^4)x^1 \\
&&& +a^4bx^0 \\
(m^4) * (n^2) &= && x^6 \\
&&& +(2b+4a)x^5 \\
&&& +(b^2+8ab+6a^2)x^4 \\
&&& +(4ab^2+12a^2b+4a^3)x^3 \\
&&& +(6a^2b^2+8a^3b+a^4)x^2 \\
&&& +(4a^3b^2+2a^4b)x^1 \\
&&& +a^4b^2x^0 \\
(m^4) * (n^3) &= && x^7 \\
&&& +(3b+4a)x^6 \\
&&& +(3b^2+12ab+6a^2)x^5 \\
&&& +(b^3+12ab^2+18a^2b+4a^3)x^4 \\
&&& +(4ab^3+18a^2b^2+12a^3b+a^4)x^3 \\
&&& +(6a^2b^3+12a^3b^2+3a^4b)x^2 \\
&&& +(4a^3b^3+3a^4b^2)x^1 \\
&&& +a^4b^3x^0 \\
(m^4) * (n^4) &= && x^8 \\
&&& +(4b+4a)x^7 \\
&&& +(6b+16ab+6a)x^6 \\
&&& +(4b+24ab+24ab+4a)x^5 \\
&&& +(b+16ab+36ab+16ab+a)x^4 \\
&&& +(4ab+24ab+24ab+4ab)x^3 \\
&&& +(6ab+16ab+6ab)x^2 \\
&&& +(4ab+4ab)x^1 \\
&&& +abx^0 \\
(m^4) * (n^5) &= && x^9 \\
&&& +(5b+4a)x^8 \\
&&& +(10b^2+20ab+6a^2)x^7 \\
&&& +(10b^3+40ab^2+30a^2b+4a^3)x^6 \\
&&& +(5b^4+40ab^3+60a^2b^2+20a^3b+a^4)x^5 \\
&&& +(b^5+20ab^4+60a^2b^3+40a^3b^2+5a^4b)x^4 \\
&&& +(4ab^5+30a^2b^4+40a^3b^3+10a^4b^2)x^3 \\
&&& +(6a^2b^5+20a^3b^4+10a^4b^3)x^2 \\
&&& +(4a^3b^5+5a^4b^4)x^1 \\
&&& +a^4b^5x^0
\end{aligned}$$

1.7 step 6 $(m^5) * (n^i)$

$$\begin{aligned}(m^5) * (n^0) &= \\(m^5) * (n^1) &= \\(m^5) * (n^2) &= \\(m^5) * (n^3) &= \\(m^5) * (n^4) &= \\(m^5) * (n^5) &= \end{aligned}$$

2 Groups

Notes from the meeting:

$$G = \langle a, b; [a, b]^n = 1 \rangle$$

The dihedral group:

$$D = \langle a, b; a^2 = 1, b^{2n} = 1 \rangle$$

$$is|D| = 4n?$$

G maps to D under ρ

$$a \rightarrow a$$

$$b \rightarrow b$$

$$[a, b] = a^{-1}b^{-1}ab$$

$$\rho([a, b]) = b^2$$

$$\rho([a, b]^n) = b^{2n}$$

So ρ is a homomorphism. Let $K = \text{kernel } \rho$ and $G/K \cong D$. There is an algorithm to rewrite elements in K in terms of coset representatives. For example, let

$$\alpha = (a^{-1}b^{-1}ab)^n$$

3 Series

3.1 Generalized Binomial Expansion

From [1] we read:

By 1665, Issac Newton had found a simple way to expand – his word was “reduce” – binomial expressions into series. For him, such reductions would be a means of recasting binomials in alternate form as well as an entryway into the method of fluxions. This theorem was the starting point for much of Newton’s mathematical innovation.

As described in the *epistola prior*, the issue at hand was to reduce the binomial $(P + PQ)^{m/n}$ and to do so whether m/n “is integral or (so to speak) fractional, whether positive or negative”. . . .

Newton discovered a pattern for expanding not only elementary binomials like $(1+x)^5$ but more sophisticated ones like $\frac{1}{\sqrt[3]{(1+x)^5}} = (1+x)^{5/3}$. The reduction, as Newton explained to Leibniz, obeyed the rule

$$(P + PQ)^{m/n} = p^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \text{etc.},$$

where each of A, B, C, \dots represents the previous term, as will be illustrated below. This is his famous binomial expansion, although perhaps in an unfamiliar guise.

Newton provided the example of $\sqrt{c^2 + x^2} = [c^2 + c^2(x^2)/c^2]^{1/2}$. Here, $P = c^2$, $Q = x^2/c^2$, $m = 1$, and $n = 2$. Thus,

$$\sqrt{c^2 + x^2} = (c^2)^{1/2} + \frac{1}{2}A\frac{x^2}{c^2} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \dots$$

To identify A, B, C , and the rest, we recall that each is the immediately preceding term. Thus, $A = (c^2)^{1/2} = c$, giving us

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \dots$$

Likewise B is the previous term – i.e., $B = x^2/2c$ – so at this stage we have

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \dots$$

The analogous substitutions yield $C = -x^4/8c^3$ and then $D = x^6/16c^5$. Working from left to right in this fashion, Newton arrived at

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} - \frac{x^6}{16c^5} - \frac{5x^8}{128c^7}$$

Obviously, the technique has a recursive flavor: one finds the coefficient of x^8 from the coefficient of x^6 , which in turn requires the coefficient of x^4 , and so on. Although the modern reader is probably accustomed to a “direct” statement of the binomial theorem, Newton’s recursion has an undeniable appeal, for it streamlines the arithmetic when calculating a numerical coefficient from its predecessor.

For the record, it is a simple matter to replace A, B, C, \dots by their equivalent expressions in terms of P and Q , then factor the common $P^{m/n}$ from both sides to arrive at the result found in today’s texts:

$$(1 + Q)^{m/n} = 1 + \frac{m}{n}Q + \frac{\frac{m}{n}(\frac{m}{n} - 1)}{2 \times 1}Q^2 + \frac{\frac{m}{n}(\frac{m}{n} - 1)(\frac{m}{n} - 2)}{3 \times 2 \times 1}Q^3$$

Newton likened such reductions to the conversion of square roots into infinite decimals, and he was not shy in touting the benefits of the operation. “It is a convenience attending infinite series,” he wrote in 1671,

that all kinds of complicated terms, . . . may be reduced to the class of simple quantities, i.e., to an infinite series of fractions whose numerators and denominators are simple terms, which will thus be freed from those difficulties that in their original form seem'd almost insuperable

To be sure, freeing mathematics from insuperable difficulties is a worthy undertaking.

One additional example may be helpful. Consider the expansion of $1/\sqrt{1-x^2}$, which Newton put to good use in a result we shall discuss later in the chapter. We first write this as $(1-x^2)^{-1/2}$, identify $m = -1$, $n = 2$, and $Q = -x^2$, to get

$$\begin{aligned} \frac{1}{\sqrt{1-x^2}} &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{(-1/2)(-3/2)}{2 \times 1}(-x^2)^2 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)}{3 \times 2 \times 1}(-x^2)^3 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)(-7/2)}{4 \times 3 \times 2 \times 1}(-x^2)^4 + \dots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \dots \end{aligned}$$

3.2 Inverting Series

Having described a method for reducing certain binomials to infinite series of the form $z = A + Bx + Cx^2 + Dx^3 + \dots$, Newton next sought a way of finding the series for x in terms of z . In modern terminology, he was seeking the inverse relationship. The resulting technique involves a bit of heavy algebraic lifting, but it warrents our attention for it too will appear later on. As Newton did, we describe the inversion procedure by means of a specific example.

Beginning with the series $z = x - x^2 + x^3 - x^4 + \dots$, we rewrite it as

$$(x - x^2 + x^3 - x^4 + \dots) - z = 0$$

and discard all powers of x greater than or equal to the quadratic. This, of course, leaves $x - z = 0$, and so the inverted series begins as $x = z$.

Newton was aware that discarding all those higher degree terms rendered the solution inexact. The exact answer would have the form $x = z + p$, where p is a series yet to be determined. Substituting $z + p$ for x gives

$$[(z + p) - (z + p)^2 + (z + p)^3 - (z + p)^4 + \dots]$$

which we then expand and rearrange to get

$$\begin{aligned} &[-z^2 + z^3 - z^4 + z^5 - \dots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \dots]p \\ &+ [-1 + 3z - 6z^2 + 10z^3 - \dots]p^2 + [1 - 4z + 10z^2 - \dots]p^3 \end{aligned}$$

$$+[-1 + 5z - \dots]p^4 + \dots = 0$$

Next, jettison the quadratic, cubic, and higher degree terms in p and solve to get

$$p = \frac{z^2 - z^3 + z^4 - z^5 + \dots}{1 - 2z + 3z^2 - 4z^3 + \dots}$$

Newton now did a second round of weeding, as he tossed out all but the lowest power of z in the numerator and denominator. Hence p is approximately $z^2/1$, so the inverted series at this stage looks like $x = z + p = z + z^2$.

But p is not exactly z^2 . Rather, we say that $p = z^2 + q$, where q is a series to be determined. To do so, we substitute to get

$$\begin{aligned} &[-z^2 + z^3 - z^4 + z^5 - \dots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \dots](z^2 + q) \\ &+ [-1 + 3z - 6z^2 + 10z^3 - \dots](z^2 + q)^2 + [1 - 4z + 10z^2 - \dots](z^2 + q)^3 \\ &+ [-1 + 5z - \dots](z^2 + q)^4 + \dots = 0 \end{aligned}$$

We expand and collect terms by powers of q :

$$\begin{aligned} &[-z^3 + z^4 - z^6 + \dots] + [1 - 2z + z^2 + 2z^3 - \dots]q \\ &+ [-1 + 3z - 3z^2 - 2z^3 + \dots]q^2 + \dots \end{aligned}$$

As before, discard terms involving powers of q above the first, solve to get

$$q = \frac{z^3 - z^4 + z^6 - \dots}{1 - 2z + z^2 + 2z^3 + \dots}$$

and then drop all but the lowest degree terms top and bottom to arrive at $q = z^3/1$. At this point, the series looks like $x = z + z^2 + q = z + z^2 + z^3$.

The process would be continued by substituting $q = z^3 + r$. Newton, who had a remarkable tolerance for algebraic monotony, seemed able to continue such calculations *ad infinitum* (almost). But eventually even he was ready to step back, examine the output, and seek a pattern. Newton put it this way: "Let it be observed here, by the bye, that when 5 or 6 terms ... are known, they may be continued at pleasure for most part, by observing the analogy of the progression"

For our example, such an examination suggests that

$$x = z + z^2 + z^3 + z^4 + z^5 + \dots$$

is the inverse of the series

$$z = x - x^2 + x^3 - x^4 + \dots$$

with which we began.

In what sense can this be trusted? After all, Newton discarded most of his terms most of the time, so what confidence remains that the answer is correct?

Again, we take comfort in the following “check.” The original series

$$z = x - x^2 + x^3 - x^4 + \dots$$

is geometric with common ratio $-x$, and so in closed form $z = x/(1 + x)$. Consequently, $x = z/(1 - z)$, which we recognize to be the sum of the geometric series

$$z + z^2 + z^3 + z^4 + z^5 + \dots$$

. This is precisely the result to which Newton’s procedure has led us. Everything seems to be in working order.

4 Riedemeister-Schrier

References

- [1] A Calculus Gallery
- [2] William Sit paper
- [3] Axiom, The 30 Year Horizon